The Roll Model of Trade Prices

Christopher Ting

http://cting.x10host.com/TMU/TMU.html

http://www.mysmu.edu/faculty/christophert/
✉: christophert@smu.edu.sg

June 13, 2018

Broad Lesson Plan

What is Transaction Cost?

- [Stigler, 1964] Cost of consummating a transaction
- [Demsetz, 1968] Cost of exchanging ownership title
- Brokerage fee plus the bid-ask spread
- Assuming market efficiency, [Roll, 1984] defines and derives an effective bid-ask spread, which is equal to twice the first-order serial covariance of price changes.

$$c = 2\sqrt{-\mathbb{C}\left[\Delta p_i\,,\,\Delta p_{i-1}\right]}.$$

Assumptions Underlying Quotes-Based Measures

1. Designated market makers and investors who submit limit orders are capable of setting bid and ask quotes such that the efficient price falls within the quotes.
2. The bid and ask quotes are the only available prices in the market.
3. The mid price of the quotes equals the efficient price.

Conceptual Framework

- Unobservable true price or efficient price S_i
- Observable transaction price P_i
- Observable trade sign Q_i
- Postulated relation between $s_i := \ln S_i$ and $p_i := \ln P_i$

$$p_i = s_i + \frac{c}{2} Q_i + u_i, \tag{1}$$

where the parameter c is the pricing error of Roll (1984), which is also being interpreted as the **implicit** or **liquidity cost of trading** in percent.

- u_i is i.i.d. white noise and co-varies with neither s_i nor Q_i.

What is Trade Direction Q_i?

- Buyer-initiated trade: Trade occurs at the ask price, $Q_i = 1$.
- Seller-initiated trade: Trade occurs at the bid price, $Q_i = -1$.
- More generally, trade occurs higher (lower) than the mid price is buyer-initiated (seller-initiated) trade.
- First-order difference operator: $\Delta Q_i := Q_i - Q_{i-1}$
- Possible values of ΔQ_i
  - 0 $Q_i Q_{i-1} = 1$
  - 1 $Q_i = 1, Q_{i-1} = 0$ or $Q_i = 0, Q_{i-1} = -1$
  - -1 $Q_i = -1, Q_{i-1} = 0$ or $Q_i = 0, Q_{i-1} = 1$
  - 2 $Q_i = 1, Q_{i-1} = -1$
  - -2 $Q_i = -1, Q_{i-1} = 1$

Geometric Brownian Motion

- The unobservable log efficient price s_i is assumed to follow the geometric Brownian motion with μ being the drift rate and σ the volatility (diffusion coefficient).
- The efficient price S_i at time t_i when a trade is executed as

$$\begin{aligned} S_i &= S_0 \exp\left(\mu\, t_i + \sigma W_i\right) \\ &= S_{i-1} \exp\left(\mu\, \Delta t_i + \sigma \Delta W_i\right). \end{aligned} \tag{2}$$

- Thus, given the duration Δt_i, the variance of the Brownian motion W_i is $\sigma^2 \Delta t_i$.
- The stochastic process W_i is such that $W_0 = 0$ and $\Delta W_i = \sqrt{\Delta t_i}\, \psi_i$, where ψ_i is a standard normal random variable.

Log Trade Price Difference

- Applying Itô's formula, equation (2) becomes

$$s_i = s_{i-1} + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t_i + \sigma \Delta W_i. \tag{3}$$

- By taking the difference $\Delta p_i = p_i - p_{i-1}$ and from equation (3), the following is obtained:

$$\Delta p_i = \frac{c}{2}\Delta Q_i + a\,\Delta t_i + \psi_i\sqrt{\Delta t_i} + u_i\,, \tag{4}$$

where $a = \mu - \frac{1}{2}\sigma^2$, which is the drift rate of the log efficient price.

Additional Assumptions for Derivation

A. The random variable ψ_i is independently and identically distributed with zero mean and unit variance. Namely, $\mathbb{C}\left[\psi_i\,,\,\psi_j\right] = 0$ if $i \neq j$ and 1 otherwise.

B. The random variable ψ_i is independent of the duration Δt_j, and of the change in trade sign ΔQ_j, for all i and j.

C. The duration Δt_i and the change in trade sign ΔQ_j are independent, for all i and j. Namely, $\mathbb{C}\left[\Delta t_i\,,\,\Delta Q_j\right] = 0$ for all i and j.

D. The covariance between the current trade sign Q_i and the previous trade sign Q_{i-1} is stationary. In other words, $\mathbb{C}\left[Q_i\,,\,Q_{i-1}\right] = \mathbb{C}\left[Q_{i-1}\,,\,Q_{i-2}\right]$.

Covariance of Log Price Change

- The main construct in Roll's model revolves around the covariance of price change Δp_i with its first lag, Δp_{i-1}.
- From equation (4) and with these two assumptions, we obtain

$$\begin{aligned}\mathbb{C}\left[\Delta p_i\,,\,\Delta p_{i-1}\right] &= a^2\,\mathbb{C}\left[\Delta t_i\,,\,\Delta t_{i-1}\right] + \frac{c^2}{4}\,\mathbb{C}\left[\Delta Q_i\,,\,\Delta Q_{i-1}\right] \\ &\quad + \frac{ac}{2}\,\mathbb{C}\left[\Delta t_i\,,\,\Delta Q_{i-1}\right] + \frac{ac}{2}\,\mathbb{C}\left[\Delta t_{i-1}\,,\,\Delta Q_i\right] \\ &= a^2\,\mathbb{C}\left[\Delta t_i\,,\,\Delta t_{i-1}\right] + \frac{c^2}{4}\,\mathbb{C}\left[\Delta Q_i\,,\,\Delta Q_{i-1}\right].\end{aligned}$$

Roll's Model as a Special Case

- Let $a = 0$.
- Compute the covariance by substituting in equation (4) and obtain

$$\mathbb{C}\left[\Delta p_i\,,\,\Delta p_{i-1}\right] = \frac{c^2}{4}\,\mathbb{C}\left[\Delta Q_i\,,\,\Delta Q_{i-1}\right]. \tag{5}$$

- By expanding the covariance on the right hand side, and assuming that $\mathbb{C}(Q_i\,,\,Q_j) = 0$ if $i \neq j$, then

$$\mathbb{C}\left[\Delta p_i\,,\,\Delta p_{i-1}\right] = -\frac{c^2}{4}\,\mathbb{V}\left[Q_{i-1}\right], \tag{6}$$

- $Q_i = \pm 1$ being equally likely as assumed in Roll's paper, the variance of the trade sign is one. Roll's formula ensues:

$$c = 2\sqrt{-\,\mathbb{C}\left[\Delta p_i\,,\,\Delta p_{i-1}\right]}. \tag{7}$$

A More General Roll's Model

- To simplify the notations, we write

$$\begin{aligned} \mathbb{C}\left[\Delta p_i\,,\,\Delta p_{i-1}\right] &= \rho_p\,\sigma_p^2\,, \\ \mathbb{C}\left[\Delta t_i\,,\,\Delta t_{i-1}\right] &= \rho_t\,\sigma_t^2\,, \\ \mathbb{C}\left[\Delta Q_i\,,\,\Delta Q_{i-1}\right] &= \rho_q\,\sigma_q^2\,, \end{aligned}$$

  where ρ_p, ρ_t, and ρ_q are the first-order correlations of the price change Δp_i, the duration Δt_i, and the change in trade sign ΔQ_i, respectively. The corresponding variances are σ_p^2, σ_t^2, and σ_q^2.

- Hence, we obtain

$$c = \frac{2}{\sigma_q}\sqrt{\frac{\rho_p\,\sigma_p^2 - a^2\rho_t\,\sigma_t^2}{\rho_q}}. \tag{8}$$

A Property

- To proceed further, let $\rho_\star$ be the first-order auto-correlation of the trade sign Q_i and $\sigma_\star^2$ the variance.
- Then in view of Assumption D,

$$
\begin{aligned}
\mathbb{C}\left[\Delta Q_i\,,\,\Delta Q_{i-1}\right] &= \mathbb{C}\left[Q_i - Q_{i-1}\,,\,Q_{i-1} - Q_{i-2}\right] \\
&= 2\,\mathbb{C}\left[Q_i\,,\,Q_{i-1}\right] - \sigma_\star^2 - \mathbb{C}\left[Q_i\,,\,Q_{i-2}\right] \\
&= 2\,\mathbb{C}\left[Q_i\,,\,Q_{i-1}\right] - \sigma_\star^2 - \rho_\star\,\mathbb{C}\left[Q_{i-1}\,,\,Q_{i-2}\right] \\
&= (2-\rho_\star)\,\mathbb{C}\left[Q_i\,,\,Q_{i-1}\right] - \sigma_\star^2 \\
&= (2-\rho_\star)\,\rho_\star\,\sigma_\star^2 - \sigma_\star^2 \\
&= -\sigma_\star^2(1-\rho_\star)^2.
\end{aligned}
$$

- Hence, the covariance of the change in trade sign with its first lag is always negative.

A Formula for General c

- In other words
$$\sigma_q^2 \rho_q = -\sigma_\star^2 (1-\rho_\star)^2.$$
- Substitute it into equation (8), we obtain
$$c = \frac{2}{\sigma_\star(1-\rho_\star)} \sqrt{a^2 \rho_t \, \sigma_t^2 - \rho_p \, \sigma_p^2}\,. \tag{9}$$
- The trading cost in percent c is now a function of
  - ★ variance $\sigma_\star^2$
  - ★ correlation $\rho_\star$ of the trade sign
  - ★ the drift rate a,
  - ★ variance σ_t^2
  - ★ correlation ρ_t of the trade duration
  - ★ variance σ_p^2
  - ★ correlation ρ_p of the price change

Other Models

- Choi, Salandro and Shastri [Choi et al., 1988]: a special case of $a = 0$ and $\sigma_\star = 1$

$$c = \frac{2}{1 - \rho_\star} \sqrt{- \mathbb{C} \left[\Delta p_i \, , \, \Delta p_{i-1} \right]} \, .$$

- George, Kaul and Nimalendran [George et al., 1991]

$$c = \frac{2}{\sqrt{\theta}} \sqrt{a^2 \rho_t \, \sigma_t^2 - \rho_p \, \sigma_p^2},$$

where θ is the unobservable proportion of the quoted spread due to order-processing costs.

Econometric Specification

- Change in trade sign ΔQ_i and Δt_i as explanatory variables
- The econometric specification is a linear regression model:

$$\Delta p_i = c_0 + \frac{c}{2}\Delta Q_i + a\Delta t_i + e_i\,. \tag{10}$$

- For ease of appreciation for the economic magnitude of the liquidity cost, we consider the following specification:

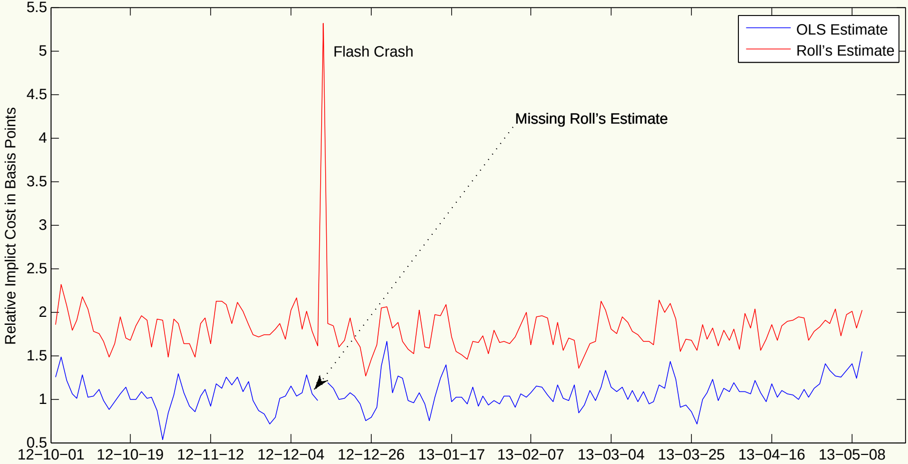
$$\Delta P_i = D_0 + \frac{D}{2}\Delta Q_i + b\Delta t_i + \epsilon_i\,. \tag{11}$$

- c is in percent, D in dollars.

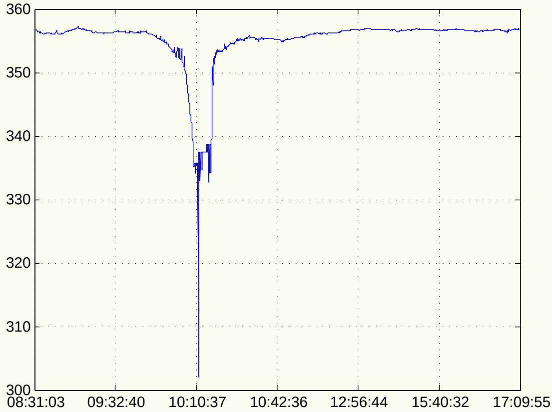
Data

- October 2012 through early May 2013
- Tick-by-tick data from Bloomberg
- Singapore futures (Simsci)
- EWS (iShares MSCI Singapore Capped ETF)

Liquidity Cost Estimates for SG Futures in SGD



Flash Crash of SG Futures on December 12, 2012



Empirical Analysis: SG Futures

	mean	std	min	5th	10th	50th	90th	95th	max
SG Futures Price	361.3	13.0	330.4	338.6	342.7	365.1	373.3	379.0	388.4
Number of Trades	4,147	1,266	1,220	1,805	2,405	4,128	5,632	6,270	8,223
OLS D (S$)	13.09	2.33	9.09	10.92	11.25	12.83	14.76	15.44	35.96
Roll's C_R (S$)	7.72	1.26	3.62	5.71	6.27	7.53	9.14	10.12	12.02
OLS c (bps)	1.81	0.34	1.27	1.50	1.56	1.79	2.04	2.12	5.31
Roll's c_R (bps)	1.07	0.16	0.53	0.83	0.89	1.06	1.26	1.37	1.66
t Statistic of c	40.8	10.8	4.3	23.5	29.5	39.9	55.0	59.1	71.2
Adjusted R^2 (%)	43.8	8.2	1.1	31.5	34.8	44.1	53.8	56.8	63.2
DW Statistic	2.24	0.07	1.84	2.12	2.16	2.25	2.32	2.34	2.42

Empirical Analysis: EWS

	mean	std	min	5th	10th	median	90th	95th	max
EWS Price (US$)	11.65	0.56	10.75	10.94	11.00	11.47	12.56	12.63	12.81
Number of Trades	1,182	487	336	540	669	1,096	1,871	2,346	2,639
OLS D (US$)	0.71	0.07	0.54	0.60	0.62	0.72	0.79	0.81	1.02
Roll's C_R (US$)	0.45	0.08	0.30	0.34	0.35	0.43	0.54	0.57	0.91
OLS c (bps)	5.20	0.50	3.90	4.44	4.63	5.23	5.71	6.03	7.60
Roll's c_R (bps)	3.26	0.58	2.19	2.48	2.58	3.19	3.95	4.11	6.79
t Statistic	26.2	9.3	10.0	14.4	15.1	25.3	38.1	39.9	72.7
Adjusted R^2 (%)	57.6	10.6	30.3	37.3	43.6	59.1	68.9	70.6	82.4
DW Statistic	2.34	0.13	2.05	2.14	2.19	2.33	2.51	2.56	2.79

Comparison of Liquidity Cost Ratios

$$\lambda = \frac{D}{\text{minimum tick size}}.$$

Exchange Name	Exchange Code	Number of Trades	Liquidity Cost Rato λ	OLS c Estimate (bps)
Singapore Exchange		4,147	0.66	1.81
FINRA-ADF	D	1,182	0.71	5.20
Direct Edge	J	350	0.58	4.20
NYSE Arca	P	731	0.48	3.48
Nasdaq	T	372	0.31	2.28
BATS Y-Exchange	Y	243	0.57	4.18
BATS Z-Exchange	Z	480	0.33	2.41

Takeaways

- Concept of true price or efficient price S_t
- Transaction price $\neq$ true price
- Concept of implicit trading cost
- Complex number may occur when using a "square-root" formula to estimate the implicit trading cost.
- Roll's formula underestimates the implicit trading cost.
- The OLS approach is more robust and general; the "square-root" formulas are special cases.

References

- Choi, J. Y., Salandro, D., and Shastri, K. (1988).
  On the estimation of bid-ask spreads: Theory and evidence.
  Journal of Financial and Quantitative Analysis, 23:219–230.
- Demsetz, H. (1968).
  The cost of transacting.
  Quarterly Journal of Economics, 82:33–53.
- George, T., Kaul, G., and Nimalendran, M. (1991).
  Estimation of the bid-ask spread and its components: A new approach.
  Review of Financial Studies, 4:623–656.
- Roll, R. (1984).
  A simple implicit measure of the effective bid-ask spread in an efficient market.
  Journal of Finance, 39:1127–1139.

References (Cont'd)

- Stigler, G. J. (1964).
  Public regulation of the securities markets.
  Journal of Business, 37:117–142.